ON CLASSES OF STRATEGIES IN DIFFERENTIAL GAMES OF EVASION OF CONTACT

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The game problem of the guiding of the controlled motion onto a given set is analyzed. The chances of the second player (the pursued) under a choice of strategies from the class of discontinuous controls formed on the feedback principle are investigated. For this purpose there are introduced into consideration the absorption sets defined for the classes of continuous, programed, and discontinuous controls, and their coincidence is proved. It is ascertained that the solution of the optimal deviation problem, which exists in the class of mixed and approximate strategies, may not necessarily exist in the class of discontinuous strategies formalized whithin the scope of the theory of differential equations in contingencies. The contents of this paper border on the investigations in [1-7].

1. Let the controlled motion be described by the equation

$$dx / dt = A(t)x + B(t)u - C(t)v \tag{1.1}$$

where x is an n-dimensional system phase vector; u and v are the control vectors of the first and second players; $A(t)$, $B(t)$, $C(t)$ are continuous matrix-valued functions of appropriate dimensions. In the phase space X_n we are given a closed convex set M onto which the first player strives to guide system (1.1), while the second player, to the contrary, is not interested in realizing the condition $x[t] \in M$. It is assumed that the realizations of the players' controls are subject to the conditions

$$u[t] \in P, \qquad v[t] \in Q \tag{1.2}$$

where P and Q are closed and bounded covex sets in the corresponding vector spaces. We shall assume that each player does not know the future behavior of his opponent but does know the position $\{t, x[t]\}$ realized at each current instant t. The initial position $\{t_0, x_0\}$ is given.

The classes of mixed and approximate strategies of the first and second players were determined in [5, 6]. It was shown that the classes of these strategies will be complete. An alternative [6] is valid for the class of mixed strategies, which in the case being considered can be stated in the following way.

Let $\{t_0, x_0\}$ be some initial position of the game, M be some closed set in the space of vectors $\{x\}$. By $\mathbf{U}^{(c)}$ and $\mathbf{V}^{(c)}$ we denote the classes of mixed strategies of the first and second players. One of the following two assertions is valid.

(1) Either there exist an instant $\vartheta \geqslant t_0$ and a strategy $U_* \in \mathbf{U}^{(c)}$ such that whatever be the strategy $V \in \mathbf{V}^{(c)}$, any motion $x[t] = x[t; t_0, x_0, U_*, V]$ hits on M not later than at the instant $t = \vartheta$.

(2) Or for any arbitrarily large instant $\vartheta > t_0$ we can find a number $\varepsilon > 0$ and a strategy $V_* \in \mathbf{V}^{(c)}$ such that whatever be the strategy $U \in \mathbf{U}^{(c)}$, the condition $x[t] \notin M^\varepsilon$ for $t_0 \leqslant t \leqslant \vartheta$ will be fulfilled for any motion $x[t] = x[t;\ t_0,\ x_0,\ U,\ V_*]$.

Here $x[t;\ t_0,\ x_0,\ U,\ V]$ are motions generated by strategies U, V and satisfying the initial condition $x[t_0] = x_0$; the symbol M^ε denotes a closed ε-neighborhood of set M.

Thus, the positional game problem posed above is always solvable in the class of mixed strategies, i. e. either we can find in the class $\mathbf{U}^{(c)}$ a strategy of the first player, guaranteeing the contact of the point $x[t]$ with the set M, or among the strategies $V \in \mathbf{V}^{(c)}$ there exists a strategy for evasion. An analogous situation holds also for the class of approximate strategies [5].

In this paper we examine wider classes of players' strategies – classes of continuous strategies and classes of discontinuous controls formalized within the framework of differential equations in contingencies [8, 9]. It is shown that these classes of strategies are not complete, i. e. the alternative stated above is not true for them. An example is presented where there exists an evasion strategy V_* belonging to the class $\mathbf{V}^{(c)}$, which cannot be approximated by continuous strategies and by discontinuous controls determined in accordance with the tools of differential equations in contingencies.

Let us describe the classes of players' strategies to be considered and let us determine the absorption sets corresponding to them. An investigation of the relations between these absorption sets allows us to establish the fact that the solution of the evasion problem may not necessarily exist in the strategy classes being considered. By $\mathbf{U}_1$ $(\mathbf{V}_1)$ we denote the first (second) player's set of programed controls, i. e. the set of arbitrary measurable vector-valued functions $u(t)$ $(v(t))$ satisfying constraints (1.2) for almost all $t \geqslant t_0$.

As the second strategy class $\mathbf{U}_2$ $(\mathbf{V}_2)$ we choose the set of continuous vector-valued functions $u = u(t,\ x)$ $(v = v(t,\ x))$ satisfying the condition $u(t,\ x) \in P$ $(v(t,\ x) \in Q)$ for all $\{t,\ x\}$.

We specify the strategy class $\mathbf{V}^*$ in the following manner. Let $V = V(t,\ x)$ be a function which associates a closed convex set $V(t,\ x) \in Q$ with each position $\{t,\ x\}$, such that as $\{t,\ x\}$ changes the set $V(t,\ x)$ varies upper-semicontinuously with respect to inclusion, namely: for any sequence $\{t_k,\ x_k\}$ converging to some point $\{t_*,\ x_*\}$, an arbitrary convergent sequence v_k from the corresponding sets $V(t_k,\ x_k)$ has a limit point element from $V(t_*,\ x_*)$.

In the case when the first player acts in accordance to the program $u(\cdot) \in \mathbf{U}_1$ while the second player chooses a control, being guided by a strategy $V = V(t,\ x)$ from class $\mathbf{V}^*$, by a motion $x[t]$ of system (1.1) we shall mean every absolutely continuous function $x[t]$ which passes through the point x_0 at the instant t_0 and whose derivative satisfies almost everywhere the conditions

$$dx[t]/dt = A(t)x[t] + B(t)u(t) - C(t)v[t], \qquad v[t] \in V(t, x[t]) \quad (1.3)$$

The existence of such a function $x[t]$ follows from the theory of differential equations in contingencies [8, 9]. Thus, the third class $\mathbf{V}^*$ of strategies will comprise all possible functions $V = V(t,\ x)$ of the form described above. We remark that this class contains the second player's discontinuous controls $v = v(t,\ x)$ formed on the feedback principle; here, the motions generated by such controls are determined in accordance with the theory of differential equations with a discontinuous right-hand side [8].

Using the strategy classes $\mathbf{U}_1$, $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}^*$ introduced above, we construct the following absorption sets [5–7] in the phase space X_n.

We define the set $W_1(\tau, \vartheta)$ of programed absorption as the collection of all points $w \in X_n$, each of which possesses the property: for any programed control $v(\cdot) \in \mathbf{V}_1$ of the second player there exists a control $u(\cdot) \in \mathbf{U}_1$ of the first player, such that under the action of this pair $\{u(t), v(t)\}$, $\tau \leqslant t \leqslant \vartheta$, system (1.1) passes from the state $x(\tau) = w$ to the state $x(\vartheta) \in M$.

Analogously we construct the set $W_2(\tau, \vartheta)$ – the set of points w from X_n satisfying the condition: for any control $v(\cdot) \in \mathbf{V}_2$ of the second player we can choose a control $u(\cdot) \in \mathbf{U}_1$ of the first player, such that among the solutions $x[t]$ of system (1.1), generated by the pair $\{u(t), v(t, x)\}$, we can find a motion issuing from the point $x[\tau] = w$ and hitting on M at the instant ϑ.

We define the set $W^*(\tau, \vartheta)$ as the collection of points $w \in X_n$ such that whatever be the strategy $V(\cdot) \in \mathbf{V}^*$, there exists a control $u(\cdot) \in \mathbf{U}_1$ which generates a family of motions of system (1.1), containing the trajectory $x[t]$ satisfying the conditions $x[\tau] = w$, $x[\vartheta] \in M$.

In [7] it was shown that the equality

$$W_1(\tau, \vartheta) = W_2(\tau, \vartheta) \tag{1.4}$$

is valid for the sets constructed above.

In this paper we shall show that

$$W_1(\tau, \vartheta) = W_2(\tau, \vartheta) = W^*(\tau, \vartheta)$$

This signifies the following. If the inclusion $x_0 \in W_1(t_0, \vartheta)$ is valid for the initial position $\{t_0, x_0\}$ of the game, i.e. if the second player cannot prevent system (1.1) from being guided onto the set M at the instant ϑ by using the programed controls $v(\cdot) \in \in \mathbf{V}_1$, then he also cannot guarantee himself that M will not be hit on by choosing strategies from class $\mathbf{V}_2$ or $\mathbf{V}^*$. Conversely, the relation $x_0 \notin W^*(t_0, \vartheta)$ signifies that we can point out for the second player not only a strategy $V_0(\cdot) \in \mathbf{V}^*$ which guarantees him that system (1.1) does not hit on the aim set M at the instant ϑ, but also that there exists a programed control $v_0(\cdot) \in \mathbf{V}_1$ which in pair with any $u(\cdot) \in \mathbf{U}_1$ generates a motion $x[t]$, $t_0 \leqslant t \leqslant \vartheta$, of system (1.1), satisfying the conditions $x[t_0] = x_0$, $x[\vartheta] \notin M$.

2. Theorem. The sets $W_1(\tau, \vartheta)$, $W_2(\tau, \vartheta)$, $W^*(\tau, \vartheta)$ coincide, i.e.,

$$W_1(\tau, \vartheta) = W_2(\tau, \vartheta) = W^*(\tau, \vartheta) \qquad (-\infty < \tau \leqslant \vartheta < \infty) \tag{2.1}$$

Proof. The inclusion $W_2(\tau, \vartheta) \supset W^*(\tau, \vartheta)$ is obvious since the relation $\mathbf{V}_2 \subset \mathbf{V}^*$ is valid for the strategy calsses $\mathbf{V}_2$ and $\mathbf{V}^*$. Then, the inclusion $W_1(\tau, \vartheta) \supset W^*(\tau, \vartheta)$ holds by virtue of (1.4).

Let us show that the inverse inclusion $W_1(\tau, \vartheta) \subset W^*(\tau, \vartheta)$ also is fulfilled. We assume the contrary. Let there exist τ and ϑ $(-\infty < \tau \leqslant \vartheta < \infty)$ such that the following relation

$$W^*(\tau, \vartheta) \not\supset W_1(\tau, \vartheta) \tag{2.2}$$

is fulfilled for the corresponding sets $W_1(\tau, \vartheta)$ and $W^*(\tau, \vartheta)$, i.e. there exists a point w_* for which the relations $w_* \in W_1(\tau, \vartheta)$, $w_* \notin W^*(\tau, \vartheta)$ are fulfilled. This means the following. If at the instant τ system (1.1) is at the point w_*, then for any programed control of the second player the first player can always choose a control $u(\cdot) \in \mathbf{U}_1$ such that under the action of this pair the motion $x[t]$ of system (1.1) passes from the position $\{\tau, w_*\}$ to the point $x[\vartheta] \in M$ at the instant ϑ. However, there exists for the second player a strategy $V^\circ = V^\circ(t, x)$ from the class V* such that all motions of system (1.1), generated by the pairs $\{u(\cdot), V^\circ(\cdot)\}$ and issuing from the point $x[\tau] = w_*$, do not hit on the set M at the instant ϑ.

In order to obtain a contradiction refuting (2.2) we carry out the following construction. With each element $v(t)$, $\tau \leqslant t \leqslant \vartheta$ of the set $\mathbf{V}_1$ we associate a control $u(t)$, $\tau \leqslant t \leqslant \vartheta$, $u(\cdot) \in \mathbf{U}_1$ such that the control pair $\{u(\cdot), v(\cdot)\}$ would carry system (1.1) from the state $x[\tau] = w_*$ into the state $x[\vartheta] \in M$. We denote the set of all controls $u(\cdot) \in \mathbf{U}_1$ chosen in this manner for a given $v(\cdot) \in \mathbf{V}_1$ by $\mathbf{U}_1(v(\cdot))$. The sets $\mathbf{U}_1(v(\cdot))$ constructed are nonempty by virtue of the relation $w_* \in W_1(\tau, \vartheta)$. Let us note here that the sets $\mathbf{U}_1$, $\mathbf{V}_1$, as well as $\mathbf{U}_1(v(\cdot))$ are bounded, closed and convex in $L_2[\tau, \vartheta]$ as a consequence of the closedness and convexity of the sets M, P, Q and of the linearity of system (1.1).

Let us consider a set S consisting of all possible pairs $\{u(\cdot), v(\cdot)\}$, where $v(\cdot)$ ranges over the strategy set $\mathbf{V}_1$, while $u(\cdot)$ is chosen from the corresponding sets $\mathbf{U}_1(v(\cdot))$. The nonemptiness of set S is an obvious consequence of the nonemptiness of the sets $\mathbf{U}_1(v(\cdot))$. The convexity of S also is obvious.

We introduce into consideration a Banach space $B[\tau, \vartheta]$, whose elements are the vector-valued functions $\{u(t), v(t)\}$, $\tau \leqslant t \leqslant \vartheta$, with components from $L_2[\tau, \vartheta]$. We define the norm of the elements in $B[\tau, \vartheta]$ by the relation

$$\gamma[u, v] = \left(\int_\tau^\vartheta \left[\sum_i u_i^2(t) + \sum_j v_j^2(t)\right] dt\right)^{1/2}$$

In the space $B[\tau, \vartheta]$ the set S is bounded and weakly closed by virtue of the boundedness and weak closedness of the sets $\mathbf{U}_1$, $\mathbf{V}_1$ in $L_2[\tau, \vartheta]$ and of the closedness of M in the phase space X_n. We construct a mapping of the set S into itself in the following way. Each pair $\{u(\cdot), v(\cdot)\} \in S$ generates a motion $x[t]$, $\tau \leqslant t \leqslant \vartheta$ of system (1.1), satisfying the conditions $x[\tau] = w_*$, $x[\vartheta] \in M$. The strategy $V^\circ = V^\circ(t, x)$, mentioned above in the explanation of relation (2.2), associates the sets $V^\circ[t] = V^\circ(t, x[t])$ with this motion. Thus we have a mapping of the pair $\{u(t), v(t)\}$ from S into a collection of sets $V^\circ[t]$ $(\tau \leqslant t \leqslant \vartheta)$. With the collection of sets $V^\circ[t]$, $\tau \leqslant t \leqslant \vartheta$, we associate further the set of functions $\psi(\cdot) \in L_2[\tau, \vartheta]$ satisfying almost everywhere on $[\tau, \vartheta]$ the following condition:

$$\psi[t] \in V^\circ[t] \tag{2.3}$$

Because the sets $V^\circ(t, x)$ are semicontinuous in $\{t, x\}$, it follows from the theory of differential equations in contingencies [8, 9] that the set of functions $\psi(\cdot) \in L_2[\tau, \vartheta]$, satisfying condition (2.3), is not empty. Using each of these functions $\psi(\cdot)$ we select a function $\varphi(\cdot) \in \mathbf{U}_1(\psi(\cdot))$, i.e. in such a way that the control pair $\{\varphi(\cdot), \psi(\cdot)\}$ would take system (1.1) from the state $x[\tau] = w_*$ into $x[\vartheta] \in M$. This can be done since $w_* \in W_1(\tau, \vartheta)$. By Φ we denote the set of all possible pairs $\{\varphi(\cdot), \psi(\cdot)\}$ associated in this manner with the system of sets $V^\circ[t]$, $\tau \leqslant t \leqslant \vartheta$. By construction the system of sets $V^\circ[t]$, $\tau \leqslant t \leqslant \vartheta$, corresponds to some completely determined pair $\{u(\cdot), v(\cdot)\} \in S$ and, hence, the set Φ, corresponding to $V^\circ[\cdot]$, also depends on this pair. We shall note this fact by writing u and v as arguments of the set Φ, i.e. $\Phi = \Phi(u, v)$.

In passing we note the obvious fact that $\Phi(u, v) \subset S$. Thus, we have constructed a mapping of the set S into itself.

In what follows our problem will be to prove the existence of a fixed point of this mapping, namely, of a pair $\{u_*(\cdot), v_*(\cdot)\} \in S$ for which the relation

$$\{u_*(\cdot), v_*(\cdot)\} \in \Phi(u_*, v_*) \tag{2.4}$$

is valid. The presence of a fixed point will signify that, on the one hand, the motion $x_*[t]$, $\tau \leqslant t \leqslant \vartheta$ of system (1.1), corresponding to the control pair $\{u_*(\cdot), v_*(\cdot)\}$ satisfies the conditions $x[\tau] = w_*$ and $x[\vartheta] \in M$. On the other hand, this same motion $x_*[\cdot]$ by the definition of the set $\Phi(u_*, v_*)$ corresponds to the strategy pair $\{u_*(\cdot), V^\circ(\cdot)\}$ and, therefore, by virtue of the relation $w_* \notin W^*(\tau, \vartheta)$ and by the definition of the strategy $V^\circ(\cdot) \in \mathbf{V}^*$, should not hit onto the set M at the instant ϑ. We thus arrive at a contradiction with assumption (2.2) and, consequently, we prove equality (2.1).

To prove the existence of a fixed point of the mapping constructed above, we make use of a theorem of S. Karlin and H. F. Bohnenblust ([10], p. 496). Let us verify the fulfillment of its hypotheses:

1) the nonemptiness of $\Phi(u, v)$ has been shown above;

2) the convexity of $\Phi(u, v)$ is a consequence of the convexity of the sets $\mathbf{U}_1$, $V(t, x)$, M and of the linearity of system (1.1);

3) the union of all sets $\Phi(u, v)$ over $\{u(\cdot), v(\cdot)\} \in S$ is contained in the set S which possesses the property of sequential ω-compactness, i.e. in every sequence of points of S there is contained a subsequence converging weakly to some point of this set S (this property follows from the sequential ω-compactness of the classes $\mathbf{V}_1$ and $\mathbf{U}_1$ and from the closedness of M in X_n);

4) let us show the upper semicontinuity of the mapping $\{u(\cdot), v(\cdot)\}$ in $\Phi(u, v)$. We take an arbitrary sequence of points $\{u_k(\cdot), v_k(\cdot)\}$ converging weakly to $\{u_*(\cdot), v_*(\cdot)\}$ as $k \to \infty$, and also a certain sequence $\{\varphi_k(\cdot), \psi_k(\cdot)\}$

$$\{\varphi_k(\cdot), \psi_k(\cdot)\} \in \Phi(u_k, v_k) \qquad (k = 1, 2, \ldots) \tag{2.5}$$

converging weakly to $\{\varphi_*(\cdot), \psi_*(\cdot)\}$. We need to show the validity of the inclusion

$$\{\varphi_*(\cdot), \psi_*(\cdot)\} \in \Phi(u_*, v_*) \tag{2.6}$$

By virtue of the sequential ω-compactness of the set S the weak limits of the elements of S also belong to this set, i.e. $\{\varphi_*(\cdot), \psi_*(\cdot)\} \in S$. But then, from the method of construction of S and $\Phi(u_*, v_*)$ it follows that to prove (2.6) it remains to verify the validity of the inclusion

$$\psi_*(t) \in V^\circ(t, x_*[t]) = V_*^\circ[t] \tag{2.7}$$

which should be fulfilled for almost all $t \in [\tau, \vartheta]$. Here $x_*[t]$ is the motion of system (1.1) under the action of the control pair $\{u_*(\cdot), v_*(\cdot)\}$ on the interval $[\tau, \vartheta]$ with the initial condition $x[\tau] = w_*$. For a summable function $\psi_*(t)$, $\tau \leqslant t \leqslant \vartheta$, almost all points of the interval $[\tau, \vartheta]$ are Lebesgue points, i.e. the relation

$$\psi_*(t) = \lim \frac{1}{\Delta} \int_t^{t+\Delta} \psi_*(\zeta)\, d\zeta \quad \text{as} \quad \Delta \to 0 \tag{2.8}$$

is fulfilled at them. Let us show that relation (2.7) is fulfilled for the function $\psi_*(\cdot)$ at all its Lebesgue points, i.e. (2.7) is not fulfilled other than on a set of measure zero.

Let $t_* \in [\tau, \vartheta]$ be some Lebesgue point of the function $\psi_*(\cdot)$. We specify a certain number $\varepsilon > 0$. Now, for the given ε we select, by virtue of (2.8), a number $\Delta_1 > 0$ in such a way that the inequality

$$\left\| \psi_*(t_*) - \frac{1}{\Delta} \int_{t_*}^{t_*+\Delta} \psi_*(\zeta)\, d\zeta \right\| < \varepsilon \tag{2.9}$$

holds for all $\Delta \leqslant \Delta_1$. From the weak convergence in space $B[\tau, \vartheta]$ of the vector-valued functions $\{\varphi_k(\cdot), \psi_k(\cdot)\}$ follows the weak convergence of their components in $L_2[\tau, \vartheta]$. Therefore, from the convergence of the sequence $\{\varphi_k(\cdot), \psi_k(\cdot)\}$ to the point $\{\varphi_*(\cdot), \psi_*(\cdot)\}$ follows the convergence in the weak topology in spaces $L_2[\tau, \vartheta]$ of the sequences $\{\varphi_k(\cdot)\}$ and $\{\psi_k(\cdot)\}$ to the points $\varphi_*(\cdot)$ and $\psi_*(\cdot)$, respectively.

For almost all $t \in [\tau, \vartheta]$ the relation

$$\psi_k(t) \in V^\circ(t, x_k[t]) = V_k^\circ[t]$$

is valid for the functions $\psi_k(t)$, where $x_k[t]$ are the motions of system (1.1) from the initial state $x_k[\tau] = w_*$ generated by the controls $\{u_k(\cdot), v_k(\cdot)\}$. Note that for each $t \in [\tau, \vartheta]$ the sequence $\{x_k[t]\}$ converges to $x_*[t]$, where $x_*[t]$, $\tau \leqslant t \leqslant \vartheta$ is the trajectory of system (1.1) corresponding to the pair $\{u_*(\cdot), v_*(\cdot)\}$ and to the initial condition $x_*[\tau] = w_*$. Furthermore, the set of these trajectories is uniformly bounded and equicontinuous, and, therefore, from the sequence $\{x_k[\cdot]\}$ we can pick out a subsequence converging uniformly to $x_*[\cdot]$. In order not to complicate the notation, in what follows we shall take the sequence $\{x_k[\cdot]\}$ to be uniformly convergent to $x_*[\cdot]$. Using this property, as well as the upper semicontinuity of $V^\circ(t, x)$ with respect to the collection $\{t, x\}$, for the t_* and $\varepsilon > 0$ chosen above we can find a number $\Delta_2 > 0$ and a number K_1 such that the inclusion

$$V_k^\circ[t] \subset V_*^{\circ(\varepsilon)}[t_*] \tag{2.10}$$

where $V_*^{\circ(\varepsilon)}$ is a closed convex ε-neighborhood of the set V°_*, is fulfilled for all t, $|t - t_*| \leqslant \Delta_2$ and $k \geqslant K_1$.

We introduce into consideration the following auxiliary functions on $[t_*, \vartheta]$:

$$y_k(t) = \int_{t_*}^{t} \psi_k(\zeta)\, d\zeta, \qquad y_*(t) = \int_{t_*}^{t} \psi_*(\zeta)\, d\zeta$$

The sequence $y_k(t)$ converges to $y_*(t)$ as $k \to \infty$, pointwise on $[t_*, \vartheta]$. It is easy to show that the set of these absolutely continuous functions is uniformly bounded and equicontinuous, and hence, from the sequence $\{y_k(\cdot)\}$ we can pick out a subsequence converging uniformly to $y_*(\cdot)$ as $k \to \infty$. Once again, with a view to simplify the writing, we do not renotate the subsequence picked out, but we shall assume in the subsequent arguments that from the very first there holds the property: from the weak convergence of $\psi_k(\cdot)$ to $\psi_*(\cdot)$ follows the uniform convergence of $y_k(\cdot)$ to $y_*(\cdot)$. But then, for ε and $\Delta = \min\{\Delta_1, \Delta_2\}$ we can choose a number $K_2 \geqslant K_1$ such that for all $k \geqslant K_2$ and for all $t \in [t_*, \vartheta]$ there holds

$$\| y_k(t) - y_*(t) \| \leqslant {}^1\!/_2\, \varepsilon\Delta \tag{2.11}$$

Now, by virtue of (2.9) and (2.11) the following estimates

$$\left\| \psi_*(t_*) - \frac{1}{\Delta} \int_{t_*}^{t_*+\Delta} \psi_k(\zeta)\, d\zeta \right\| = \left\| \psi_*(t_*) - \frac{y_k(t_*+\Delta) - y_k(t_*)}{\Delta} \right\| \leqslant$$

$$\leqslant \left\| \psi_*(t_*) - \frac{1}{\Delta} \int_{t_*}^{t_*+\Delta} \psi_*(\zeta)\, d\zeta \right\| + \left\| \frac{y_*(t_*+\Delta) - y_k(t_*+\Delta)}{\Delta} \right\| +$$

$$+ \left\| \frac{y_*(t_*) - y_k(t_*)}{\Delta} \right\| \leqslant 2\varepsilon \tag{2.12}$$

are valid for any $k \geqslant K_2$ and $\Delta = \min\{\Delta_1, \Delta_2\}$. Moreover, (2.10) holds for almost all ζ, $|\zeta - t_*| \leqslant \Delta \leqslant \Delta_1$ and for $k \geqslant K_2 \geqslant K_1$. Consequently, the inclusion

$$\frac{1}{\Delta}\int_{t_*}^{t_*+\Delta} \varphi_k(\zeta)\, d\zeta \in V_*^{\circ(\varepsilon)}[t_*]$$

is valid by virtue of the convexity of the set $V_*^\circ[t_*]$. Hence, by virtue of (2.12), follows

$$\psi_*(t_*) \in V_*^{\circ(3\varepsilon)}[t_*] \qquad (2.13)$$

In the arguments we have made ε and t_* were chosen arbitrarily; consequently, the validity of inclusion (2.7) almost everywhere on $[\tau, \vartheta]$, which is what we had to prove, follows from condition (2.13). Thus, all the hypotheses of the theorem on the existence of a fixed point have been fulfilled and, by the same token, the original assertion (2.1) has been proved.

3. The coincidence of the absorption sets $W_1(\tau, \vartheta)$, $W_2(\tau, \vartheta)$ and $W^*(\tau, \vartheta)$ signifies that the solution of the evasion problem, in general, may not necessarily exist in the class of continuous strategies $v = v(t, x)$ and in the class of discontinuous strategies $V = V(t, x)$ formalizable within the framework of the theory of differential equations in contingencies. As an example illustrating this fact, here, as also in [7], we can cite the problem of the pursuit of an inertialess point by a material point.

By y we denote a two-dimensional vector whose components are the differences of the corresponding coordinates of the material and the inertialess points, and by z, the velocity vector of the material point. We obtain the following equations of motion:

$$dy_1 / dt = z_1 - v_1, \quad dz_1 / dt = u_1; \qquad dy_2 / dt = z_2 - v_2, \quad dz_2 / dt = u_2 \qquad (3.1)$$

where the controls satisfy the inequalities

$$u_1^2 + u_2^2 \leqslant \mu^2, \qquad v_1^2 + v_2^2 \leqslant \nu^2$$

The set M is specified by the condition $y_1 = y_2 = 0$.

It is known [1] that in the given problem, for any initial game position $x[t_0] = \{y[t_0], z[t_0]\} = x_0$ there exists a programed absorption instant $\vartheta^\circ(x_0)$, i.e. there exists the smallest value of the parameter $\vartheta = \vartheta^\circ(x_0)$ for which the inclusion

$$x_0 \in W_1(t_0, \vartheta) \qquad (3.2)$$

is fulfilled. From the equalities (2.1) proved above, from inclusion (3.2), and from the definition of the sets $W_1(\tau, \vartheta)$, $W_2(\tau, \vartheta)$, $W^*(\tau, \vartheta)$ it follows that when the second player selects any strategy from the class $\mathbf{V}_1$, $\mathbf{V}_2$ or $\mathbf{V}^*$ it is impossible to avoid contact during the time interval $[t_0, \vartheta^\circ(x_0)]$. It is also known [3, 7] that in the given problem there exist methods for forming the second player's control, which guarantee an evasion from hitting onto M during as large a time interval as desired. Such a strategy, for example, is the mixed strategy $V^e = V^e(t, x)$ extremal to some system of strongly v-stable sets [5]

$$S(t, \vartheta_*), \quad S(t, \vartheta_*) \cap M = \varnothing, \quad t_0 \leqslant t \leqslant \vartheta_* \qquad (3.3)$$

Here we can choose the number ϑ_* arbitrarily large, in particular, we can assume that $\vartheta_* > \vartheta^\circ(x_0)$.

Thus, in the example being considered the evasion strategy V is contained in the class of mixed strategies but does not belong to the strategy class $\mathbf{V}^*$. Note that from the relations $V^e \in \mathbf{V}^{(c)}$, $V^e \notin \mathbf{V}^*$ follows the existence of pieces of a nonsmooth concave boundary for the strongly v-stable sets $S(t, \vartheta_*)$ in (3.3). Indeed, if the sets $S(t, \vartheta_*)$ in (3.3) were not to have pieces of a nonsmooth concave boundary, then the extremal strategy V^e would belong to the class $\mathbf{V}^*$. (The validity of the last assertion ensues directly from the relations giving the function $V^e = V^e(t, x)$ [5]).

The example cited shows that the solution of the evasion problem exists only in the complete classes of mixed or approximate strategies and is not contained in the classes $\mathbf{V}_2$ and $\mathbf{V}^*$. This fact should be taken into account when constructing strategies which are stable to the measurement errors in the current game position $p[t] = \{t, x[t]\}$. It can be shown that in a discrete scheme for realizing such strategies the step-size in the formation of the piecewise-constant controls should be chosen not less than some positive number determined by the magnitude of the admissible measurement errors.

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